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THERMAL STRESSES IN COMPOSITE BEAMS*

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Abstract—Elementary beam theory including the effects of temperature is extended to composite beams of rectangular cross section. The validity of the theory and an estimate of the error in its use are established; the error is small for beams with nearly axial fiber reinforcement. Bounds on the thermal stress at each point and in each material are determined in a form especially useful for beams with a large number of uniformly distributed fibers.

INTRODUCTION

THIS study considers the pointwise determination of stresses arising from mechanical and thermal loading on a composite material. The objective is the determination of the validity of expressions derived according to elementary theories when the material properties are represented by discontinuous functions of position, as is the case in composites. In particular, the present analysis treats the fundamental problem of a composite beam of rectangular cross section which deforms in a plane under axial force, bending moment and temperature. The materials are assumed to be elastic with properties unaffected by temperature and bonded to one another at all interfaces.

In the first part of the study, the elementary beam theory in which plane sections remain plane is extended to composite materials. Preliminary considerations of the applicability of this theory lead to a further restriction of the analysis to fiber reinforced materials. A method of successive approximation of the solution according to the two-dimensional theory of elasticity is then employed to establish the validity of this result and to obtain an estimate of the error in its use.

The second part considers a beam to which the elementary theory is applicable, namely, one in which the reinforcing fibers do not deviate greatly from the axial direction, under an arbitrary temperature distribution. Pointwise bounds on the stress in a cross section and overall bounds on the stress in each material are determined. For a case of practical interest where there are a large number of thin fibers uniformly distributed in the beam, these bounds are especially useful since they depend only on the extreme values of the parameter αET in the section and not on the details of the distribution of temperature and materials.

ELEMENTARY BEAM THEORY

Consider a thin rectangular beam, Fig. 1, in which the coefficient of thermal expansion $\alpha(x, y)$ and the modulus $E(x, y)$ may vary from point to point in the beam. The temperature at any point is $T(x, y)$. At any section, let the origin $y = 0$ be chosen such that

$$\int_A Ey\,\mathrm{d}A = 0 \tag{1}$$

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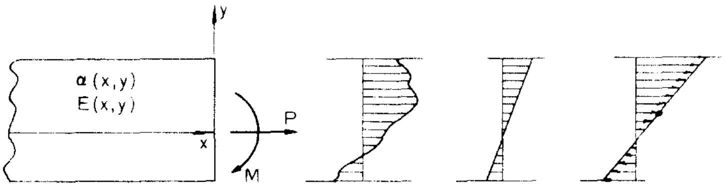


Fig. 1.

where A is the cross sectional area, and let the stress resultants be M and P. According to the elementary beam theory in which cross sections remain plane under deformation, the axial strain is a linear function of y,

$$\varepsilon = c_1 + c_2 y . \tag{2}$$

The axial stress is then

$$\sigma = E\varepsilon - \alpha ET \tag{3}$$

which gives the stress resultants

$$P = \int_A \sigma \, \mathrm{d}A$$
$$M = \int_A y\sigma \, \mathrm{d}A \tag{4}$$

if c_1 and c_2 are suitably chosen. Then with the aid of equation (1), the axial stress according to the elementary beam theory becomes

$$\sigma = -\alpha ET + \left[\frac{P+P_T}{A\bar{E}}\right]E + \left[\frac{M+M_T}{\overline{EI}}\right]Ey \tag{5}$$

where

$$P_T = \int_A \alpha ET \, \mathrm{d}A$$
$$M_T = \int_A \alpha ETy \, \mathrm{d}A \tag{6}$$

are called the thermal force and thermal moment, respectively.

The form of equation (5) is similar to that for a beam of homogeneous material, differing only by the presence of effective extensional and flexural stiffnesses

$$A\bar{E} = \int_A E \, \mathrm{d}A$$
$$\overline{EI} = \int_A Ey^2 \, \mathrm{d}A . \tag{7}$$

The stiffness $A\bar{E}$ has been written in terms of an effective modulus $\bar{E}$ which is the usual average modulus according to the "law of mixtures". The exact solution for an axially reinforced cylinder under axial load [1] leads to this same effective modulus and one may expect therefore that equation (5) will also be valid for a beam with axial reinforcing fibers. The analysis which follows is intended not only to establish the validity of the elementary theory, equation (5), for this case, but also to determine the error in applying this result when the geometry deviates from axial alignment of the reinforcement.

To this end, a more comprehensive formulation of the problem on the basis of a stress function satisfying the two-dimensional equations of elasticity is employed. A solution is then sought in the form of a series, the first term of which corresponds to the solution by the elementary theory, while subsequent terms give corrections or estimates of the error in the elementary theory. Such a procedure was first proposed in [2] and subsequently employed in [3] and [4] to establish the range of validity of the elementary beam theory for various conditions of loading (thermal and mechanical) and geometry of homogeneous beams. An analysis of those results is useful in arriving at a formulation which isolates the essential features of the present problem:

It has been shown that the elementary theory can be used with good accuracy if a homogeneous beam of small aspect ratio (i.e. ratio of depth to length) complies with the following conditions:

(a) the applied moment and axial force are smooth functions of the axial coordinate [3],
(b) the temperature is a smooth function of the axial coordinate [2],
(c) deviations from uniform depth are small and smooth along the span [4].

An examination of the development in [2] and [3] indicates that conditions (a) and (b) must be imposed on non-homogeneous beams as well. Therefore they are assumed to hold in the present problem and, in particular, the analysis will assume that P and M are constants and that $T = T(y)$ is a function of y alone. Condition (c) is significant in the present problem since one may interpret the variation in geometry along the span as resulting from the non-uniform distribution of materials in the composite. Thus the intuitive extension of condition (c) is that the elementary theory is applicable when the geometry is nearly uniform from section to section, which is the case mentioned previously of nearly axial reinforcing fibers.

The formulation according to the two-dimensional theory of elasticity is carried out for the member shown in Fig. 2, in which both the aspect ratio and the ratio of fiber thickness to the depth of the member are assumed to be small, i.e.

$$\frac{c}{L} \ll 1, \qquad \frac{w}{c} \ll 1. \tag{8}$$

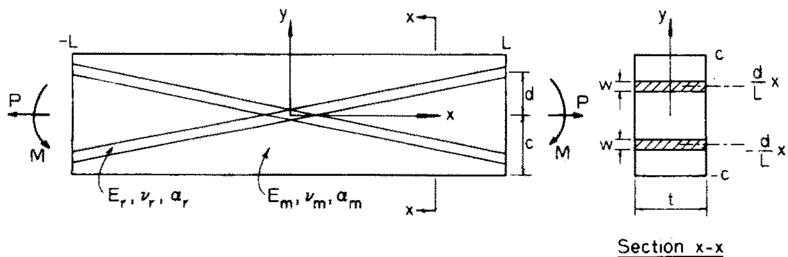


FIG. 2.

Consequently, the intersection of the fibers in the vicinity of the point $x = y = 0$ covers a very small region in the member and need not be specified in detail. It will also suffice to state the boundary conditions at $x = \pm L$ in terms of stress resultants alone, since according to St. Venant's principle the solution at some distance from the ends will not be affected appreciably by self-equilibrating stresses on the ends. A symmetrical fiber arrangement is chosen to simplify the calculations, and the slopes of the fibers with respect to the axial direction in Fig. 2, $\pm d/L$, are shown as small quantities (i.e. $d < c$) in anticipation of such a restriction on the applicability of equation (5).

For a non-homogeneous but locally isotropic material, the Airy stress function in two dimensions must satisfy the equation

$$\left(\frac{1}{E}\phi_{,yy}\right)_{,yy} - \left(\frac{\nu}{E}\phi_{,xx}\right)_{,yy} + \left(\frac{1}{E}\phi_{,xx}\right)_{,xx} - \left(\frac{\nu}{E}\phi_{,yy}\right)_{,xx} + 2\left(\frac{1+\nu}{E}\phi_{,xy}\right)_{,xy} = -(\alpha T)_{,xx} - (\alpha T)_{,yy} \tag{9}$$

where a comma signifies differentiation. For the given loading on the beam, the stress function $\phi(x, y)$ is also subject to the boundary conditions

$$\begin{aligned} \phi(x, c) &= \frac{c}{t}P - \frac{M}{t} \\ \phi_{,y}(x, c) &= \frac{P}{t} \\ \phi(x, -c) &= \phi_{,y}(x, -c) = 0\,. \end{aligned} \tag{10}$$

According to the method in [2], a solution for the stress function is generated as a formal series

$$\phi = \phi_1 + \phi_2 + \phi_3 + \dots \tag{11}$$

in which successive terms satisfy the equations

$$\begin{aligned} \left(\frac{1}{E}\phi_{1,yy}\right)_{,yy} &= -(\alpha T)_{,yy} \\ \left(\frac{1}{E}\phi_{2,yy}\right)_{,yy} &= \left(\frac{\nu}{E}\phi_{1,xx}\right)_{,yy} + \left(\frac{\nu}{E}\phi_{1,yy}\right)_{,xx} - 2\left(\frac{1+\nu}{E}\phi_{1,xy}\right)_{,xy} - (\alpha T)_{,xx} \\ \left(\frac{1}{E}\phi_{i,yy}\right)_{,yy} &= \left(\frac{\nu}{E}\phi_{i-1,xx}\right)_{,yy} + \left(\frac{\nu}{E}\phi_{i-1,yy}\right)_{,xx} \\ &\quad - 2\left(\frac{1+\nu}{E}\phi_{i-1,xy}\right)_{,xy} - \left(\frac{1}{E}\phi_{i-2,xx}\right)_{,xx} \qquad i \geq 2 \end{aligned} \tag{12}$$

If the leading term ϕ_1 satisfies the boundary conditions, equations (10), and all other terms are subject to homogeneous boundary conditions, then equations (9) and (10) are satisfied formally by the series in equation (11).

The material properties appearing in these equations are discontinuous functions of x and y for the member in Fig. 2. They can be written in the form

$$\begin{aligned} E(x,y) &= E_m + (E_r - E_m)V(x,y) \\ \nu(x,y) &= \nu_m + (\nu_r - \nu_m)V(x,y) \\ \alpha(x,y) &= \alpha_m + (\alpha_r - \alpha_m)V(x,y) \end{aligned} \tag{13}$$

where

$$\begin{aligned} V(x,y) &= 1 \text{ for } (x,y) \text{ in the reinforcement} \\ &= 0 \text{ for } (x,y) \text{ in the matrix.} \end{aligned} \tag{14}$$

In terms of the unit step function

$$\begin{aligned} U(z) &= 1 \quad \text{for } z > 0 \\ &= 0 \quad \text{for } z < 0 \end{aligned} \tag{15}$$

which will be defined for four arguments, each vanishing at one fiber–matrix interface, namely

$$\begin{aligned} U_1 &= U(z_1) = U\left(y + \frac{d}{L}X + \frac{w}{2}\right) \\ U_2 &= U(z_2) = U\left(y + \frac{d}{L}X - \frac{w}{2}\right) \\ U_3 &= U(z_3) = U\left(y - \frac{d}{L}X + \frac{w}{2}\right) \\ U_4 &= U(z_4) = U\left(y - \frac{d}{L}X - \frac{w}{2}\right) \end{aligned} \tag{16}$$

the function $V(x, y)$ becomes

$$V(x,y) = U_1 - U_2 + U_3 - U_4 . \tag{17}$$

Direct integration of the first of equations (12) and application of the boundary conditions, equations (10), give the solution for the first term of the series in equation (11),

$$\phi_1 = -\int_{-c}^{y}\int_{-c}^{y} \alpha ET \,\mathrm{d}y\,\mathrm{d}y + \frac{P+P_T}{A\bar{E}}\int_{-c}^{y}\int_{-c}^{y} E \,\mathrm{d}y\,\mathrm{d}y + \frac{M+M_T}{\overline{EI}}\int_{-c}^{y}\int_{-c}^{y} Ey \,\mathrm{d}y\,\mathrm{d}y \tag{18}$$

A is the cross-sectional area $2ct$. From this stress function, the axial stress in the composite member is

$$\begin{aligned} \sigma_{1x} &= \phi_{1,yy} \\ &= -\alpha ET + \frac{P+P_T}{A\bar{E}}E + \frac{M+M_T}{\overline{EI}}Ey . \end{aligned} \tag{19}$$

It is apparent that the stress σ_{1x} given by the first term of the series is identical to that obtained by the elementary theory, equation (5). This conclusion could have been reached by considering separately a composite beam subjected to thermal and mechanical axial forces (P and P_T) alone, or under thermal and mechanical moments (M and M_T). Because the analyses for the two cases are very similar, only the former will be treated in detail hereafter, and only the conclusions will be stated for the latter. Thus in the subsequent analysis M and M_T are taken to be zero. In order to satisfy the latter condition at all sections of the beam, it is necessary, when symmetry is taken into account, that the temperature $T(y)$ be an even function of y. Then the remaining stresses from the first term of the series are

$$\begin{aligned}\tau_1 &= -\phi_{1,xy} = \int_{-c}^{y} (\alpha ET)_{,x}\,\mathrm{d}y - \frac{P+P_T}{A\bar{E}} \int_{-c}^{y} E_{,x}\,\mathrm{d}y - \frac{P_{T,x}}{A\bar{E}} \int_{-c}^{y} E\,\mathrm{d}y \\ \sigma_{1y} &= \phi_{1,xx} = -\int_{-c}^{y}\int_{-c}^{y} (\alpha ET)_{,xx}\,\mathrm{d}y\,\mathrm{d}y + \frac{P+P_T}{A\bar{E}} \int_{-c}^{y}\int_{-c}^{y} E_{,xx}\,\mathrm{d}y\,\mathrm{d}y \\ &\quad + 2\frac{P_{T,x}}{A\bar{E}} \int_{-c}^{y}\int_{-c}^{y} E_{,x}\,\mathrm{d}y\,\mathrm{d}y + \frac{P_{T,xx}}{A\bar{E}} \int_{-c}^{y}\int_{-c}^{y} E\,\mathrm{d}y\,\mathrm{d}y.\end{aligned} \tag{20}$$

These stresses are identical to those obtained using the axial stress from elementary theory in the equilibrium equations

$$\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} &= 0 \\ \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0\,.\end{aligned} \tag{21}$$

Consequently, the solution according to the elementary theory is completely contained in the first term ϕ_1 of the series, and the stresses derived from subsequent terms will give corrections to that theory.

The integrals appearing in equations (19) and (20) contain derivatives of the unit step function and are evaluated formally in Appendix A, where a catalog of reference integrations is also given for subsequent use. The detailed expressions for the stresses, equations (19) and (20), derived from ϕ_1 are written out in Appendix B, but as an example they are given here for the simple case of constant T:

$$\begin{aligned}\sigma_{1x} &= \left[(E_r - E_m)\frac{P+P_T}{A\bar{E}} - (\alpha_r E_r - \alpha_m E_m)T\right] V(x,y) + E_m \frac{P+P_T}{A\bar{E}} - \alpha_m E_m T \\ \tau_1 &= \frac{d}{L}\left[(E_r - E_m)\frac{P+P_T}{A\bar{E}} - (\alpha_r E_r - \alpha_m E_m)T\right] V(x,y)\,\mathrm{sgn}\,x\,\mathrm{sgn}\,y \\ \sigma_{1y} &= \left(\frac{d}{L}\right)^2 \left[(E_r - E_m)\frac{P+P_T}{A\bar{E}} - (\alpha_r E_r - \alpha_m E_m)T\right] V(x,y)\end{aligned} \tag{22}$$

where

$$\mathrm{sgn}\,z = U(z) - U(-z). \tag{23}$$

Note that in contrast to the result for homogeneous materials, the thermal stresses are not zero under uniform temperature except when $\alpha_m = \alpha_r$.

The function ϕ_1, equation (18), can now be used in the second of equations (12) to obtain the second term of the series. The axial stress σ_{2x} obtained from this second term

(ϕ_2) is given in Appendix B for a temperature $T(y)$, together with the dependence of the stresses τ_2 and σ_{2y} on the slope of the reinforcement, d/L. Here the result is written once again for the case of uniform temperature:

$$\begin{aligned}\sigma_{2x} &= \left(\frac{d}{L}\right)^2 B\left[\left(1-\frac{w}{c}\frac{E_r-E_m}{\bar{E}}\right)V(x,y)-\frac{w}{c}\frac{E_m}{\bar{E}}\right]\\ \tau_2 &= \left(\frac{d}{L}\right)^3 B\left(1-\frac{w}{c}\frac{E_r-E_m}{\bar{E}}\right)V(x,y)\operatorname{sgn}x\operatorname{sgn}y\\ \sigma_{2y} &= \left(\frac{d}{L}\right)^4 B\left(1-\frac{w}{c}\frac{E_r-E_m}{\bar{E}}\right)V(x,y)\end{aligned} \tag{24}$$

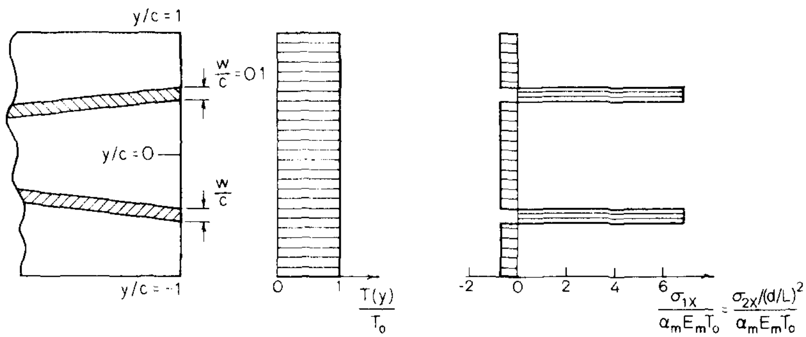
where

$$\begin{aligned}B = {} & \{E_r[\alpha_r+(1+\nu_m)\alpha_m]-(2+\nu_r)\alpha_m E_m\}T\\ & +\{(2+\nu_r)E_m-(2+\nu_m)E_r\}\frac{P+P_T}{A\bar{E}}.\end{aligned} \tag{25}$$

The values of σ_{1x} and σ_{2x} in a specific member under uniform and parabolic temperature distributions are shown in Figs. 3 and 4.

Subsequent terms of the series, equation (11), give stresses which include successively higher powers of the slope (d/L), the ith term giving

$$\begin{aligned}\sigma_{ix} &= O\left[\left(\frac{d}{L}\right)^{2(i-1)}\right]\\ \tau_i &= O\left[\left(\frac{d}{L}\right)^{2i-1}\right] \qquad i = 1, 2, 3, \ldots\\ \sigma_{iy} &= O\left[\left(\frac{d}{L}\right)^{2i}\right].\end{aligned} \tag{26}$$



$\frac{E_r}{E_m} = 50$ $\quad \frac{\alpha_m}{\alpha_r} = 10$ $\quad P = 0$

$\nu_r = 0.2$ $\quad \nu_m = 0.4$ $\quad \frac{P_T}{A\bar{E}} = 0.237\,\alpha_m T_0$

FIG. 3.

It can then be concluded that the stresses in the member in Fig. 2 under axial loading are given by the elementary theory (as expressed by equations (22)) provided the slope of the fibers is small. An estimate of the error in this result will be given by equations (24).

A similar formulation for the stresses in the member of Fig. 2 when thermal and mechanical moments (M and M_T) are present leads to the same result expressed in equations (26). Consequently, the elementary theory can be said to apply to this member under arbitrary thermal and mechanical loading, within an error of order $(d/L)^2$. In order to be of practical value, the conclusions reached here must be extended to composite beams containing many fibers. For two families of parallel fibers, symmetrically oriented with respect to the axis of the beam, the analysis follows exactly as in the case of two fibers, but the function $V(x, y)$, equation (14), is given as the sum of step functions equal in number to twice the number of fibers in a cross section. The elementary theory which gives the axial stress in equation (5) is again found to be applicable within an error of the order of the square of the slope of the fibers. Moreover, this latter result need not be restricted to two phase composites. However, the second of conditions (8) now requires the ratio of total reinforcement area to total cross sectional area to be small, so that the condition of small reinforcement volume fraction must be imposed.

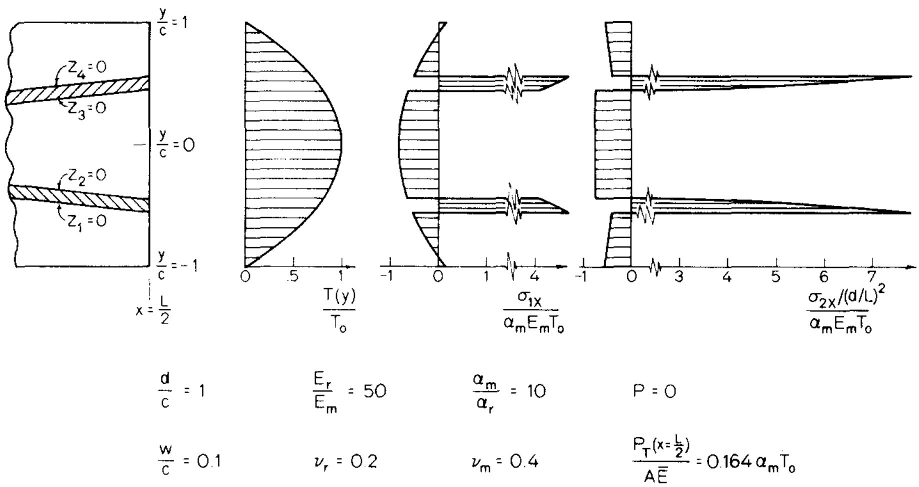


FIG. 4.

For other reinforcement geometries, it may be concluded intuitively that the elementary theory will give accurate results if the deviations from the axial direction are small, but this has not been verified in the present study. In addition, two-dimensional beam theory has been employed which implies that the reinforcement extends over the width t as shown in Fig. 2. In practice this situation may not be realized and it will be necessary to include unsymmetrical bending. The result for that case is given in [5].

It has thus been shown that the elementary beam theory may be applied to fiber-reinforced materials, not only in the obvious case of axial fibers, but also when the fibers are inclined to the axis of the composite beam. Roughly speaking, equation (26) indicates that the elementary theory, equation (5), will apply within an error of the order of 10 per cent for fiber inclinations up to 15°.

BOUNDS ON THERMAL STRESSES

Given a composite beam of rectangular cross section in which the axial stress is predicted with acceptable accuracy by the elementary theory, equation (5). It is desired to evaluate bounds on this stress

$$\sigma_m(y) \leq \sigma(y) \leq \sigma_M(y) \tag{27}$$

at any point y in a given cross section under the action of a temperature $T(x, y)$ alone (i.e. $P = M = 0$) such that the function αET is bounded as follows:

$$-\tau_m \leq \alpha ET \equiv \tau(y) \leq \tau_M. \tag{28}$$

The coordinate x is omitted from equation (27) and all subsequent equations with the understanding that a specific cross section of the beam is identified. For each of the materials in the cross section, it is also desired to obtain overall bounds on the stress in that section

$$\begin{aligned} \Sigma_m &\leq \sigma_m(y) \\ \Sigma_M &\geq \sigma_M(y). \end{aligned} \tag{29}$$

Note that there will be values of Σ_m and Σ_M for each material phase under a given temperature distribution.

For the present analysis, it is assumed that the materials are symmetrically distributed with respect to the centroid of the cross section, so that the origin $y = 0$ specified in equation (1) coincides with the centroid of the section, although the result could be generalized to exclude this restriction.

Equation (5) may be rewritten in the form

$$\sigma(\eta) = -\tau(\eta) + \frac{r_i}{2}\int_{-1}^{1} \tau(\zeta)\left[1 + \frac{\eta\zeta}{\rho^2}\right] d\zeta \tag{30}$$

where the dimensionless quantities are

$$\begin{aligned} \eta &= y/c \\ r_i &= E(\eta)/\bar{E} \\ \rho^2 &= \frac{\int_{-1}^{1} E\eta^2 \, d\eta}{\int_{-1}^{1} E \, d\eta} = \frac{\overline{EI}}{c^2 A\bar{E}} \end{aligned} \tag{31}$$

and $2c$ is the depth of the beam. The quantity r_i is a function of η but can only take on discrete values, one for each material, which is indicated by a subscript. The quantity ρ^2, when evaluated for a homogeneous beam (i.e. when $\overline{EI}/A\bar{E} = I/A$), reduces to the dimensionless radius of gyration of the section; in that case $\rho^2 = \frac{1}{3}$ and equation (30) simplifies to the analogous expression in [6]. In the more general case of composite beams,

$$0 < \rho^2 < 1. \tag{32}$$

Bounds on the stress $\sigma(\eta)$ can be constructed as in [6] and [7] by maximizing both terms in equation (30). The first term is dealt with directly by equation (28), while the integral in the second term is bounded by replacing τ in the integrand by its maximum or its minimum value, depending on the sign of the portion of the integrand in brackets throughout the range of integration $-1 \leq \zeta \leq 1$. Three cases arise, depending on the location η within the cross section of the point where the bounds on $\sigma(\eta)$ are sought:

$$-1 \leq \eta \leq -\rho^2: \quad \left[1+\frac{\eta\zeta}{\rho^2}\right] > 0 \quad -1 \leq \zeta < -\frac{\rho^2}{\eta} \tag{33a}$$
$$< 0 \quad -\frac{\rho^2}{\eta} < \zeta \leq 1$$

$$|\eta| < \rho^2; \quad \left[1+\frac{\eta\zeta}{\rho^2}\right] > 0 \quad -1 \leq \zeta \leq 1 \tag{33b}$$

$$\rho^2 \leq \eta \leq 1; \quad \left[1+\frac{\eta\zeta}{\rho^2}\right] < 0 \quad -1 \leq \zeta < -\frac{\rho^2}{\eta}$$
$$> 0 \quad -\frac{\rho^2}{\eta} < \zeta \leq 1. \tag{33c}$$

The maximum and minimum stresses at any point in the center portion of the section, equation (33b), may be determined by substituting τ_M and $-\tau_m$, for $\tau(\eta)$ in equation (30) which gives

$$\begin{aligned} \sigma_M(\eta) &= \tau_m + r_i\tau_M \\ \sigma_m(\eta) &= -\tau_M - r_i\tau_m \end{aligned} \qquad |\eta| < \rho^2 \tag{34}$$

The procedure is repeated for points in the remaining portions of the cross section, but the range of integration must be separated into the ranges specified by equation (33a) or equation (33c). After substituting the bounds on $\tau(\eta)$ and carrying out the integrations, the result may be written in the form

$$\begin{aligned} \sigma_M(\eta) &= \tau_m(1-r_i)+\frac{r_i}{4}(\tau_M+\tau_m)\left[2+\frac{|\eta|}{\rho^2}+\frac{\rho^2}{|\eta|}\right] \\ \sigma_m(\eta) &= -\tau_M(1-r_i)-\frac{r_i}{4}(\tau_M+\tau_m)\left[2+\frac{|\eta|}{\rho^2}+\frac{\rho^2}{|\eta|}\right]. \end{aligned} \qquad \rho^2 \leq |\eta| \leq 1 \tag{35}$$

For the particular case when $\rho^2 = \frac{1}{3}$ (which corresponds to $\overline{EI}/A\overline{E} = I/A$), the variation of the upper and lower bounds with location in the cross section is shown by the solid line in Fig. 5. Note that the value of σ_M or σ_m at a point η may be evaluated from Fig. 5 only when the value of r_i at that point is known—i.e. when it is known which material occupies the point in question.

When the temperature is known to be symmetric or antisymmetric, improved bounds can be obtained by first simplifying equation (30). Thus when $T(y)$ is symmetric ($M_T = 0$) the bounds become

$$\begin{aligned} \sigma_M(\eta) &= \tau_m + r_i\tau_M \\ \sigma_m(\eta) &= -\tau_M - r_i\tau_m \end{aligned} \qquad |\eta| \leq 1. \tag{36}$$

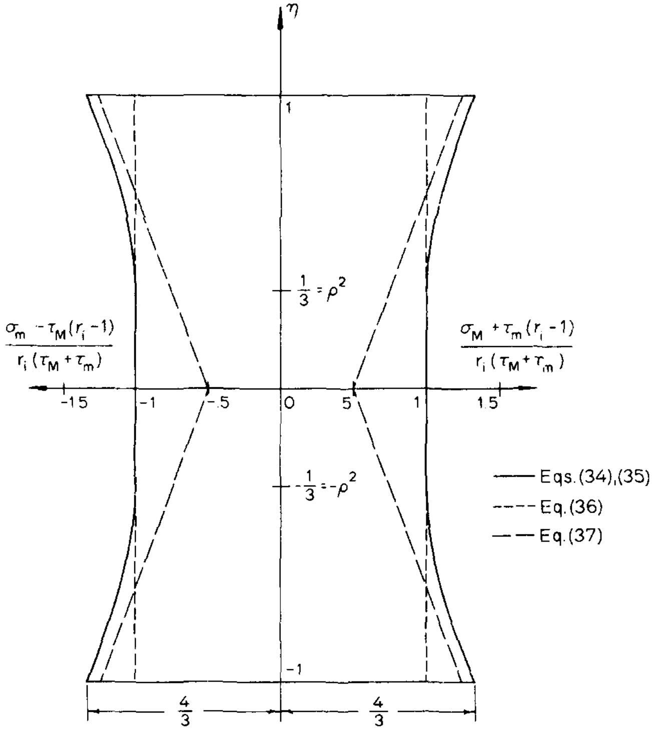


FIG. 5.

and when $T(y)$ is antisymmetric ($P_T = 0$), so that $\tau_m = \tau_M$, they are

$$\left.\begin{aligned}\sigma_M(\eta) &= \tau_m(1-r_i)+\frac{r_i}{4}(\tau_M+\tau_m)\left(2+\frac{|\eta|}{\rho^2}\right)\\ \sigma_m(\eta) &= -\tau_M(1-r_i)-\frac{r_i}{4}(\tau_M+\tau_m)\left(2+\frac{|\eta|}{\rho^2}\right).\end{aligned}\right\}\quad |\eta| \le 1 \tag{37}$$

These bounds are also shown in Fig. 5 for the case $\rho^2 = \frac{1}{3}$.

Overall bounds, equation (29), on the stress in any material in the section can be obtained by noting that the bounds $\sigma_M(\eta)$ and $\sigma_m(\eta)$ increase monotonically in magnitude with $|\eta|$. Consequently, the largest value of $|\eta|$ identified with a material r_i may be used in equations (34) or (35) to determine the overall bound for that material. Alternatively, wider bounds may be obtained more readily by evaluating equations (35) at the extreme fiber $\eta = 1$ for each value of r_i. The overall bounds become

$$\begin{aligned}\frac{\Sigma_M+\tau_m(r_i-1)}{r_i(\tau_M+\tau_m)} &= -\left[\frac{\Sigma_m-\tau_M(r_i-1)}{r_i(\tau_M+\tau_m)}\right]\\ &= \frac{1}{4}\left(2+\rho^2+\frac{1}{\rho^2}\right)\end{aligned} \tag{38}$$

which reduce to the value $\frac{4}{3}$ given in [6] and [7] for homogeneous beams ($r_i = 1$, $\rho^2 = \frac{1}{3}$). Similarly, the overall bounds corresponding to equations (36) for a symmetric temperature distribution are

$$\frac{\Sigma_M + \tau_m(r_i - 1)}{r_i(\tau_M + \tau_m)} = -\left[\frac{\Sigma_m - \tau_M(r_i - 1)}{r_i(\tau_M + \tau_m)}\right] = 1 \tag{39}$$

and from equations (37) for antisymmetric temperatures,

$$\frac{\Sigma_M + \tau_m(r_i - 1)}{r_i(\tau_M + \tau_m)} = -\left[\frac{\Sigma_m - \tau_M(r_i - 1)}{r_i(\tau_M + \tau_m)}\right] = \frac{1}{4}\left(2 + \frac{1}{\rho^2}\right). \tag{40}$$

An example of the evaluation of the bounds is given in Fig. 6 for the parabolic temperature distribution in Fig. 4, which gives

$$\begin{aligned} \tau_m &= 0 \\ \tau_M &= 0{\cdot}7975\, \alpha_r E_r T_0 . \end{aligned} \tag{41}$$

Note that τ_M is the value of τ at $|\eta| = 0{\cdot}45$ even though the maximum temperature occurs at $\eta = 0$, which illustrates the overriding effect of the material parameters relative to the temperature in identifying the values τ_M and τ_m, equation (28). In a case such as this, where the distribution of materials and temperature are known precisely, values of τ_M and τ_m might be determined for each of the materials in the cross section and the evaluation of bounds from equation (30) would then utilize this additional information. However the present simpler form of the bounds makes them especially useful for certain composites of practical interest in which the locations of reinforcing materials are not known precisely. It is only necessary that the given volume fraction of reinforcement consist of a large number of fine fibers uniformly distributed in the beam. For then, the quantity ρ^2, equation (31), approaches the value $\frac{1}{3}$ (i.e. $\overline{EI} \to \bar{E}I$), and the bounds are given in Fig. 5 without further knowledge of the locations of materials.

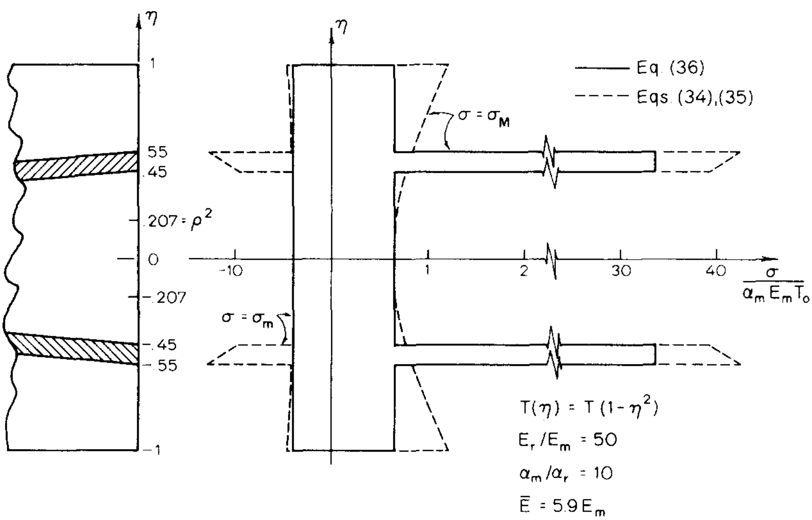


FIG. 6.

No general statement can be made concerning the proximity of the bounds to the actual stresses in a composite beam, because this will depend strongly on the temperature distribution, the distribution of materials and the relative magnitudes of the material parameters. Thus, if these were such that the quantity $\tau(y)$ were constant in the section, the bounds would coincide with the values given by the elementary theory.

In this part of the study, bounds have been derived for the stress resulting from an arbitrary temperature distribution in a composite beam to which the elementary theory is applicable. There is no restriction on the number of materials in the composite, but for convenience a symmetric distribution was considered. The bounds are given in the form of two results: firstly, the maximum stresses which may occur at any point, and secondly, the maximum stress which may occur in each material in a given cross section. These reduce to the values in [6] and [7] for homogeneous materials. A column analogy of the type developed in [7] for the calculation of the bounds could also be established here but it appears to be more useful for beams of general rather than rectangular configuration [5]. Finally, the present results can also be applied to cylindrical bending of plates by substituting $E/(1-\nu^2)$ for E, and $(1+\nu)\alpha$ for α in the final expressions.

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APPENDIX A

Integrations of the function $V(x, y)$ can be carried out without difficulty. However, there arise also integrations of derivatives of this function with respect to x and their evaluation might be effected by employing delta functions. Alternatively, the unit step functions may be approximated by

$$U(z) = \tfrac{1}{2}(1+\tanh Nz) \qquad N \gg 1 \tag{A1}$$

in evaluating the integrals and then the limit $N \to \infty$ taken. Thus if

$$\begin{aligned} z &= y+ax+b \\ z_0 &= -c+ax+b \end{aligned} \tag{A2}$$

one finds

$$\begin{aligned} \int_{-c}^{y} \frac{\partial U(z)}{\partial x}\,\mathrm{d}y &= a[U(z)-U(z_0)] \\ \int_{-c}^{y}\int_{-c}^{y} \frac{\partial^2 U(z)}{\partial x^2}\,\mathrm{d}y\,\mathrm{d}y &= a^2[U(z)-U(z_0)]. \end{aligned} \tag{A3}$$

Using equations (A3) and/or direct integration and/or integration by parts, the following integrals can be evaluated. The values given here reflect the fact that each of the arguments z_i is negative at $y = -c$ so that the step functions of equations (16) are zero at that point.

$$
\begin{aligned}
&\int_{-c}^{y} V(x, y)\,\mathrm{d}y = \sum_{i=1}^{4} (-1)^{i-1} z_i U_i \\
&\int_{-c}^{y} V_{,x}\,\mathrm{d}y = \frac{d}{L}(U_1 - U_2 - U_3 + U_4) = -\frac{d}{L} V(x, y)\,\mathrm{sgn}\,x\,\mathrm{sgn}\,y \\
&\int_{-c}^{y} V_{,x} T(y)\,\mathrm{d}y = \frac{d}{L}(U_1 T_1 - U_2 T_2 - U_3 T_3 + U_4 T_4) \\
&\int_{-c}^{y} V_{,xx} T(y)\,\mathrm{d}y = -\left(\frac{d}{L}\right)^2 \sum_{i=1}^{4} (-1)^{i-1} T_{i,y} \\
&\int_{-c}^{y}\int_{-c}^{y} V(x, y)\,\mathrm{d}y\,\mathrm{d}y = \frac{1}{2} \sum_{i=1}^{4} (-1)^{i-1} z_i^2 U_i \\
&\int_{-c}^{y}\int_{-c}^{y} V_{,x}\,\mathrm{d}y\,\mathrm{d}y = \frac{d}{L}(z_1 U_1 - z_2 U_2 - z_3 U_3 + z_4 U_4) \\
&\int_{-c}^{y}\int_{-c}^{y} V_{,xx}\,\mathrm{d}y\,\mathrm{d}y = \left(\frac{d}{L}\right)^2 V(x, y) \\
&\int_{-c}^{y}\int_{-c}^{y} V_{,xx} T(y)\,\mathrm{d}y\,\mathrm{d}y = \left(\frac{d}{L}\right)^2 \sum_{i=1}^{4} (-1)^{i-1} U_i (T_i - z_i T_{i,y}).
\end{aligned}
\qquad \text{(A4)}
$$

The notation in these expressions is

$$
\begin{aligned}
T_i &= T(y)|_{z_i = 0} \\
T_{i,y} &= T_{,y}|_{z_i = 0}
\end{aligned}
\qquad \text{(A5)}
$$

where $z_i (i = 1 \ldots 4)$ are the arguments in equations (16) of the text.

APPENDIX B

Using the results in Appendix A and the expressions in equations (13), the stresses, equations (19), obtained from the function ϕ_1 and assuming that T is a function of y only become

$$\sigma_{1x} = \left[(E_r - E_m)\frac{P+P_T}{A\bar{E}} - (\alpha_r E_r - \alpha_m E_m)T(y)\right]V(x,y) + E_m\frac{P+P_T}{A\bar{E}} - \alpha_m E_m T(y)$$

$$\frac{\tau_1}{(d/L)} = (E_r - E_m)\frac{P+P_T}{A\bar{E}}V(x,y)\operatorname{sgn} x \operatorname{sgn} y + (\alpha_r E_r - \alpha_m E_m)\left[U_1T_1 - U_2T_2 - U_3T_3 + U_4T_4 - \frac{T_1 - T_2 - T_3 + T_4}{2c\bar{E}}\left[(E_r - E_m)\sum_{i=1}^{4}(-1)^{i-1}z_iU_i + E_m(y+c)\right]\right]$$

$$\frac{\sigma_{1y}}{(d/L)^2} = (E_r - E_m)\frac{P+P_T}{A\bar{E}}V(x,y) - (\alpha_r E_r - \alpha_m E_m)\left[\sum_{i=1}^{4}(-1)^{i-1}U_i(T_i - z_iT_{i,y}) + \frac{\sum_{i=1}^{4}(-1)^{i-1}T_{i,y}}{4c\bar{E}}\left[E_m(y+c)^2 + (E_r - E_m)\sum_{i=1}^{4}(-1)^{i-1}z_i^2U_i\right] - \frac{T_1 - T_2 - T_3 + T_4}{c\bar{E}}(E_r - E_m)(z_1U_1 - z_2U_2 - z_3U_3 + z_4U_4)\right]. \tag{B1}$$

In these expressions, T_i and $T_{i,y}$ are the quantities defined in Appendix A, equations (A5).

It should be noted that these results apply when the thermal conductivities of the two materials are equal. When they are unequal, the temperature will still be continuous across interfaces between the two materials, but the derivative $T_{,y}$ will not be continuous. The derivation of equations (B1) remains the same, but the result will contain the quantities $T_{,y}$ evaluated on either side of the interfaces $z_i = 0$.

Two integrations of the second of equations (12) give the axial stress from the term ϕ_2 in the form

$$\sigma_{2x} = E(x,y)\left[\int_{-c}^{y}\int_{-c}^{y}\left(\frac{\nu}{E}\sigma_{1x}\right)_{,xx}\mathrm{d}y\,\mathrm{d}y + 2\int_{-c}^{y}\left(\frac{1+\nu}{E}\tau_1\right)_{,x}\mathrm{d}y - \int_{-c}^{y}\int_{-c}^{y}\alpha_{,xx}T\,\mathrm{d}y\,\mathrm{d}y + c_0 + c_1 y\right] + \nu(x,y)\sigma_{1y}. \tag{B2}$$

The constants of integration c_0 and c_1 are to be evaluated from the conditions

$$\int_{-c}^{c}\sigma_{2x}\,\mathrm{d}y = 0$$
$$\int_{-c}^{c}\sigma_{2x}y\,\mathrm{d}y = 0 \tag{B3}$$

and the stresses τ_2 and σ_{2y} may be determined from the equations of equilibrium once σ_{2x} is evaluated. After substituting equations (B1) and making use of the results in Appendix A, equations (B2) give

$$\sigma_{2x} = \left(\frac{d}{L}\right)^2 \left\{E(x, y)\left[B_1 + (\alpha_r E_r - \alpha_m E_m)B_2 + \frac{\alpha_r E_r - \alpha_m E_m}{c\bar{E}} B_3\right] + v(x, y)B_4\right\} \tag{B4}$$

where

$$B_1 = c_0 + c_1 y + \frac{P + P_T}{A\bar{E}}\left[(v_r - v_m) - 2(1+v_r)\frac{E_r - E_m}{E_r}\right] V(x, y)$$
$$- [(1+v_r)\alpha_r - (1+v_m)\alpha_m] \sum_{i=1}^{4} (-1)^{i-1} U_i(T_i - z_i T_{i,y})$$

$$B_2 = 2\left(\frac{1+v_r}{E_r} - \frac{1+v_m}{E_m}\right)\left[\begin{aligned} &U_1(T_1 - z_1 T_{1,y}) - U_2(T_1 - z_2 T_{1,y}) + U_3(T_3 - z_3 T_{3,y}) \\ &\quad - U_4(T_3 - z_4 T_{3,y}) \\ &\quad - U(x)U_3[T_1 - T_2 + (T_{1,y} - T_{2,y})z_3] + U(x)U_4[T_1 - T_2 \\ &\quad + (T_{1,y} - T_{2,y})z_4] \\ &\quad - U(-x)U_1[T_3 - T_4 + (T_{3,y} - T_{4,y})z_1] + U(-x)U_2[T_3 - T_4 \\ &\quad + (T_{3,y} - T_{4,y})z_2] \end{aligned}\right]$$
$$+ \frac{2(1+v_m)}{E_m} \sum_{i=1}^{4} (-1)^{i-1} U_i(T_i - z_i T_{i,y})$$

$$B_3 = \frac{1}{4}\sum_{i=1}^{4} (-1)^{i-1} T_{i,y} \left[\begin{aligned} &(2+v_m)(y+c)^2 - \left[(v_r - v_m) - 2(1+v_m)\frac{E_r - E_m}{E_m}\right] \sum_{i=1}^{4} (-1)^{i-1} z_i^2 U_i \\ &+ 4\left(\frac{1+v_r}{E_r} - \frac{1+v_m}{E_m}\right)\left[\begin{aligned} &E_m \sum_{i=1}^{4} (-1)^{i-1} z_i U_i \left(y + c - \frac{z_i}{2}\right) \\ &+ (E_r - E_m)\left[\frac{z_1^2}{2}(U_1 - U_2) + \frac{z_3^2}{2}(U_3 - U_4)\right. \\ &+ \frac{w^2}{2}(U_2 + U_4) + wU(x)(z_3 U_3 - z_4 U_4) \\ &\left. + wU(-x)(z_1 U_1 - z_2 U_2)\right] \end{aligned}\right] \end{aligned}\right]$$

$$+ (T_1 - T_2 - T_3 + T_4)\left[\begin{aligned} &\left[v_r - v_m - 2(1+v_r)\frac{E_r - E_m}{E_r}\right](z_1 U_1 - z_2 U_2 - z_3 U_3 + z_4 U_4) \\ &- \left(\frac{1+v_r}{E_r} - \frac{1+v_m}{E_m}\right)\left[\begin{aligned} &E_m[(y+c)(U_1 - U_2 - U_3 + U_4) \\ &\quad - z_1 U_1 + z_2 U_2 + z_3 U_3 - z_4 U_4] \\ &+ (E_r - E_m)w[U(-x)U_1 - U(x)U_2 \\ &\quad - U(x)U_3 + U_4(U(x) + 1)] \end{aligned}\right] \end{aligned}\right]$$

$$B_4 = (E_r - E_m)\frac{P+P_T}{A\bar{E}}V(x,y) - (\alpha_r E_r - \alpha_m E_m)\sum_{i=1}^{4}(-1)^{i-1}U_i(T_i - z_i T_{i,y})$$

$$+\frac{\alpha_r E_r - \alpha_m E_m}{c\bar{E}}\begin{bmatrix}(E_r - E_m)(T_1 - T_2 - T_3 + T_4)(z_1 U_1 - z_2 U_2 - z_3 U_3 + z_4 U_4) \\ \left[\frac{E_r - E_m}{4}\sum_{i=1}^{4}(-1)^{i-1}z_i^2 U_i + \frac{E_m}{4}(y+c)^2\right]\sum_{i=1}^{4}(-1)^{i-1}T_{i,y}\end{bmatrix} \qquad \text{(B5)}$$

The constants c_0 and c_1 in equation (B2) can be shown to be of order $(d/L)^2$ by employing equations (B3), and have accordingly been rewritten with this factor in equation (B4). Because of the complexity of equation (B4), the constants have not been evaluated explicitly (except for the case T = constant, equations (22)) but they can be determined numerically to satisfy equations (B3). For illustration, the axial stress distributions σ_{1x} and σ_{2x} are shown in Figs. 3 and 4 for constant and parabolic distributions of temperature in a member with specified geometry and materials. In the latter, the constants c_0 and c_1, equation (B4) were found to be $-0{\cdot}255\alpha_m T_0$ and $0{\cdot}307\alpha_m T_0/c$, respectively.

The stresses σ_y and τ arising from the second term (ϕ_2) of the series have not been shown explicitly. Using the equations of equilibrium and equation (B4), it can be shown that

$$\tau_2 = O\left[\left(\frac{d}{L}\right)^3\right]$$
$$\sigma_{2y} = O\left[\left(\frac{d}{L}\right)^4\right] \qquad \text{(B6)}$$

It should be emphasized that the expression for σ_{2x} has been evaluated here only to verify in Figs. 3 and 4 that the magnitude of this term is characterized by the coefficient $(d/L)^2$. The complexity of this expression would preclude its general use to obtain more accurate values of the stress.

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Абстракт—Расширяется элементарная теория балок, с учетом зффектов температуры на сложенные балки прямоугольного поперечного сечения. Определяются область важности теории и оценка погрешности. Погрешность оказывается малая для балки с почти осевым усилением волокнами. Даются пределы термических напряжений в каждой точке и каждом материале, в форме специально пригодной для балки с болршим числом одномерно расположенных волокон.